

# AN ADI CRANK-NICOLSON ORTHOGONAL SPLINE COLLOCATION METHOD FOR THE TWO-DIMENSIONAL FRACTIONAL DIFFUSION-WAVE EQUATION\*

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**Abstract.** A new method is formulated and analyzed for the approximate solution of a two-dimensional time-fractional diffusion-wave equation. In this method, orthogonal spline collocation is used for the spatial discretization and, for the time-stepping, a novel alternating direction implicit (ADI) method based on the Crank-Nicolson method combined with the  $L1$ -approximation of the time Caputo derivative of order  $\alpha \in (1, 2)$ . It is proved that this scheme is stable, and of optimal accuracy in various norms. Numerical experiments demonstrate the predicted global convergence rates and also superconvergence.

**Key words.** Two-dimensional fractional diffusion-wave equation, Caputo derivative, alternating direction implicit method, orthogonal spline collocation method, Crank-Nicolson method, stability, optimal global convergence estimates, superconvergence

**AMS subject classifications.** 65M70, 65M12, 65M15, 35R11

**1. Introduction.** In this paper, we focus on the formulation and analysis of an alternating direction implicit (ADI) orthogonal spline collocation (OSC) method for the approximate solution of the two-dimensional time-fractional diffusion-wave problem

$$(1.1) \quad {}^C_0 D_t^\alpha u(x, y, t) = \Delta u(x, y, t) + f(x, y, t), \quad (x, y, t) \in \Omega_T \equiv \Omega \times (0, T],$$

with the initial conditions

$$(1.2) \quad u(x, y, 0) = \varphi(x, y), \quad D_t u(x, y, 0) = \phi(x, y), \quad (x, y) \in \overline{\Omega} = \Omega \cup \partial\Omega,$$

and the boundary condition

$$(1.3) \quad u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T].$$

Here,  $\Delta$  is the Laplace operator,  $\Omega = (0, 1) \times (0, 1)$  with boundary  $\partial\Omega$ ,  $\varphi(x, y)$ ,  $\phi(x, y)$  and  $f(x, y, t)$  are given sufficiently smooth functions in their respective domains. Also,  ${}^C_0 D_t^\alpha u(x, y, t)$  is the Caputo fractional derivative of order  $\alpha$  ( $1 < \alpha < 2$ ) defined by

$$(1.4) \quad {}^C_0 D_t^\alpha u(x, y, t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(x, y, s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}},$$

where  $\Gamma(\cdot)$  denotes the Gamma function, named after Caputo [4] who was one of the first to use this operator in applications and to investigate some of its properties.

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Equation (1.1) is called a time-fractional partial differential equation of order  $\alpha$  since it is intermediate between the diffusion equation ( $\alpha = 1$ ) and the wave equation ( $\alpha = 2$ ). In recent years, fractional partial differential equations have gained rapidly in popularity and importance as new modeling tools in a variety of fields, such as physics, biology, mechanical engineering, environmental science, signal processing, systems identification, electrical and control theory, finance, and hydrology; see, for example, [23, 24]. In particular, the fractional diffusion-wave equation (1.1) models wave propagation in viscoelastic materials.

Several approaches have been proposed for the solution of fractional partial differential equations in one and several space variables; see, for example, [1, 7, 14, 17, 21] and references in these papers. In particular, alternating direction implicit (ADI) methods have been employed recently for the solution of multidimensional problems. ADI methods were first introduced in the context of finite difference methods (FDMs) for parabolic and elliptic problems by Peaceman and Rachford [20] in the 1950s, and such methods in conjunction with various types of spatial discretizations continue to be studied extensively today, especially for the numerical solution of time-dependent problems; see [12] and references therein. The attraction of these techniques is that they replace the solution of multidimensional problems by sequences of one-dimensional problems, thus reducing the computational cost. For solving fractional problems in two space variables, ADI methods have been employed in numerous contexts. Meerschaert et al., [16] formulated an ADI FDM based on the backward Euler method to solve a class of space-fractional partial differential equations with variable coefficients, and, for the same problem, Tadjeran and Meerschaert [22] derived an ADI method based on the Crank-Nicolson finite difference method, and used Richardson extrapolation to improve the spatial accuracy. For a space-fractional advection-dispersion equation, Chen and Liu [5] considered an ADI FDM backward Euler method and obtained second-order accuracy in both space and time on using Richardson extrapolation. Zhang and Sun [27] formulated and analyzed two ADI FDMs based on the  $L1$  approximation [18] and the backward Euler method for the time-fractional sub-diffusion equation comprising (1.1)–(1.3) with  $0 < \alpha < 1$ . These methods are proved to be second-order in space and of order  $\min(2\alpha, 2 - \alpha)$  and  $\min(1 + \alpha, 2 - \alpha)$ , respectively, in time. Zhang et al., [28] formulated and analyzed a compact ADI FDM and a Crank-Nicolson ADI FDM for the time-fractional diffusion-wave equation and proved that the methods are fourth-order accurate in space and of order  $3 - \alpha$  in time. For the same problem but with Neumann boundary conditions, Ren and Sun [21] formulated similar ADI methods of the same accuracy. Wang and Wang [25] formulated an ADI FDM for a class of space-fractional diffusion equations. They provided no analysis of the method but demonstrated its efficiency. Cui considered compact ADI FDMs for a time-fractional diffusion equation with the Riemann-Liouville fractional derivative in [6] and the Caputo derivative in [7]. ADI FDMs have also been used in the solution of three-dimensional fractional problems. In particular, Liu et al., [15] proposed such a scheme for the solution of a fractional equation governing seepage flow, and used Richardson extrapolation to improve the spatial accuracy. Also, Yu et al., [26] constructed an ADI FDM method for the fractional Bloch-Torrey equation to study anomalous diffusion in the human brain.

Orthogonal spline collocation has evolved as a valuable technique for the solution of several types of partial differential equations [3], especially in combination with ADI methods for multidimensional problems [10]. The popularity of OSC methods is due in part to their conceptual simplicity, wide applicability and ease of implementa-

tion. A well-known advantage of OSC methods over finite element Galerkin methods is that the calculation of the coefficients in the equations determining the approximate solution is very fast, since no integrals need to be evaluated or approximated. Another attractive feature of OSC methods is their superconvergence properties; see, for example, [19].

A brief outline of the remainder of this paper is as follows. In section 2, standard notation and basic lemmas are presented. The ADI OSC Crank-Nicolson method for the solution of problem (1.1)–(1.3) is formulated in section 3, followed by a stability analysis of the scheme in section 4. In section 5, we derive error estimates in the  $H^\ell$  norm,  $\ell = 0, 1, 2$ , at each time step. In section 6, we present the results of numerical experiments which support the analytical rates of convergence and exhibit superconvergence. Some concluding remarks are provided in section 7.

**2. Preliminaries.** In this section, we introduce standard notation used in the formulation of OSC methods, and basic lemmas used in their analysis.

For positive integers  $r$ ,  $N_x$ ,  $N_y$ , let  $\delta_x = \{x_i\}_{i=0}^{N_x}$  and  $\delta_y = \{y_j\}_{j=0}^{N_y}$  be two partitions of  $\bar{I} = [0, 1]$  such that

$$0 = x_0 < x_1 < \dots < x_{N_x} = 1, \quad 0 = y_0 < y_1 < \dots < y_{N_y} = 1.$$

Set

$$I_k^x = (x_{k-1}, x_k), \quad h_k^x = x_k - x_{k-1}, \quad 1 \leq k \leq N_x,$$

$$I_l^y = (y_{l-1}, y_l), \quad h_l^y = y_l - y_{l-1}, \quad 1 \leq l \leq N_y,$$

and  $h = \max \left( \max_{1 \leq k \leq N_x} h_k^x, \max_{1 \leq l \leq N_y} h_l^y \right)$ . It is assumed that the collection of partitions  $\delta = \delta_x \times \delta_y$  of  $\Omega$  is quasi-uniform.

Let  $\mathcal{M}(r, \delta_x)$  and  $\mathcal{M}(r, \delta_y)$  be the spaces of piecewise polynomials of degree  $\leq r$ ,  $r \geq 3$ , defined by

$$\mathcal{M}(r, \delta_x) = \left\{ v | v \in C^1(\bar{I}), v|_{I_k^x} \in P_r, k = 1, 2, \dots, N_x, v(0) = v(1) = 0 \right\},$$

and

$$\mathcal{M}(r, \delta_y) = \left\{ v | v \in C^1(\bar{I}), v|_{I_l^y} \in P_r, l = 1, 2, \dots, N_y, v(0) = v(1) = 0 \right\},$$

where  $P_r$  denotes the set of polynomials of degree at most  $r$ . Then we set

$$\mathcal{M}(\delta) = \mathcal{M}(r, \delta_x) \otimes \mathcal{M}(r, \delta_y),$$

the set of all functions that are finite linear combinations of products  $v^x(x)v^y(y)$ , where  $v^x \in \mathcal{M}(r, \delta_x)$  and  $v^y \in \mathcal{M}(r, \delta_y)$ , and  $\dim \mathcal{M}(\delta) = (r-1)^2 N_x N_y$ .

Let  $\{\lambda_k\}_{k=1}^{r-1}$ , with  $0 < \lambda_1 < \lambda_2 < \dots < \lambda_{r-1} < 1$ , denote the nodes of the  $(r-1)$ -point Gauss quadrature rule on the interval  $\bar{I}$  with corresponding weights  $\{\omega_k\}_{k=1}^{r-1}$ , and let  $\Lambda_x = \{\xi_{i,k}^x\}_{i,k=1}^{N_x, r-1}$  and  $\Lambda_y = \{\xi_{j,l}^y\}_{j,l=1}^{N_y, r-1}$  be the sets of Gauss points in the  $x$ - and  $y$ -directions, respectively, where

$$\xi_{i,k}^x = x_{i-1} + \lambda_k h_i^x, \quad 1 \leq k \leq r-1, \quad 1 \leq i \leq N_x,$$

and

$$\xi_{j,l}^y = y_{j-1} + \lambda_l h_j^y, \quad 1 \leq l \leq r-1, \quad 1 \leq j \leq N_y.$$

Then  $\Lambda = \{\xi | \xi = (\xi^x, \xi^y), \xi^x \in \Lambda_x, \xi^y \in \Lambda_y\}$  is the set of Gauss quadrature points in  $\Omega$ , which are the collocation points.

For  $u$  and  $v$  defined on  $\Lambda$ , we define the discrete inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|_D$  by

$$\langle u, v \rangle = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} h_i^x h_j^y \sum_{k=1}^{r-1} \sum_{l=1}^{r-1} \omega_k \omega_l (uv)(\xi_{i,k}^x, \xi_{j,l}^y), \quad \|v\|_D^2 = \langle v, v \rangle.$$

For  $\ell$  a nonnegative integer, we denote by

$$(2.1) \quad \|f\|_{H^\ell} = \left( \sum_{0 \leq \alpha_1 + \alpha_2 \leq \ell} \left\| \frac{\partial^{\alpha_1 + \alpha_2} f}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right\|^2 \right)^{\frac{1}{2}}$$

the norm on the Sobolev space  $H^\ell(\Omega)$ , where  $\|\cdot\|$  denotes the usual  $L^2$  norm, sometimes written as  $\|\cdot\|_{H^0}$  for convenience.

If  $X$  is a normed space with norm  $\|\cdot\|_X$ , then we denote by  $C([0, T], X)$  the set of functions  $f \in C(\overline{\Omega}_T) \equiv C^{0,0,0}(\overline{\Omega}_T)$  such that  $f(\cdot, t) \in X$  for  $t \in [0, T]$ , and

$$\|f\|_{C([0, T], X)} = \max_{0 \leq t \leq T} \|f(\cdot, t)\|_X < \infty.$$

Let  $C^{p,q,s}(\overline{\Omega}_T)$  denote the set of functions  $f$  such that  $\frac{\partial^{i+j+n} f}{\partial x^i \partial y^j \partial t^n}$  is continuous on  $\overline{\Omega}_T$  for  $0 \leq i \leq p$ ,  $0 \leq j \leq q$ , and  $0 \leq n \leq s$ . If  $f \in C^{p,q,s}(\overline{\Omega}_T)$ , then  $\|f\|_{C^{p,q,s}}$  is defined by

$$\|f\|_{C^{p,q,s}} = \max_{0 \leq i \leq p, 0 \leq j \leq q, 0 \leq n \leq s} \max_{(x,y,t) \in \overline{\Omega}_T} \left| \frac{\partial^{i+j+n} f}{\partial x^i \partial y^j \partial t^n} \right|.$$

Throughout the paper, we denote by  $C$  a generic positive constant that is independent of  $h$  and  $\Delta t$ , unless otherwise noted and is not necessarily the same on each occurrence. Besides, we make repeated use of the Young's inequality

$$(2.2) \quad de \leq \varepsilon d^2 + \frac{1}{4\varepsilon} e^2, \quad d, e \in \mathcal{R}, \quad \varepsilon > 0,$$

Next we present several lemmas required in the stability and convergence analyses.

LEMMA 2.1. *If  $U, V \in \mathcal{M}(\delta)$ , then the following hold:*

$$(2.3) \quad \langle -\Delta U, V \rangle = \langle U, -\Delta V \rangle,$$

[11, Eq. (3.4)];

$$(2.4) \quad \langle -\Delta U, U \rangle \geq C \|\nabla U\|^2 \geq 0,$$

[11, Eq. (3.5)];

$$(2.5) \quad |\langle \Delta U, V \rangle| \leq C \|\nabla U\| \|\nabla V\|, \quad -\langle \Delta U, V \rangle \leq C [\|\nabla U\|^2 + \|\nabla V\|^2];$$

see the proof of Lemma 3.3 in [11].

LEMMA 2.2. For  $V \in \mathcal{M}(\delta)$ ,

$$(2.6) \quad \left\langle \frac{\partial^4 V}{\partial x^2 \partial y^2}, V \right\rangle \geq \left\| \frac{\partial^2 V}{\partial x \partial y} \right\|^2,$$

[11, Lemma 3.4];

$$(2.7) \quad \left\langle \frac{\partial^4 V}{\partial x^2 \partial y^2}, -\Delta V \right\rangle \geq \left\| \frac{\partial^3 V}{\partial x^2 \partial y} \right\|^2 + \left\| \frac{\partial^3 V}{\partial x \partial y^2} \right\|^2,$$

[19, Eq. (2.31)]; and

$$(2.8) \quad \|V\|_{H^2} \leq C \|\Delta V\|_D,$$

[2, Eq. (3.20)].

**3. The ADI Crank-Nicolson OSC scheme.** Let  $\{t_n\}_{n=0}^M$  be a uniform partition of  $[0, T]$  such that  $t_n = n\Delta t$ ,  $\Delta t = T/M$ , where  $M$  is a positive integer and  $\Delta t$  is the time step size. We set  $t_{n-1/2} = (n - 1/2)\Delta t$ ,  $1 \leq n \leq M$ . Next, we introduce the following notation:

$$V^n(\cdot, \cdot) = V(\cdot, \cdot, t_n), \quad 0 \leq n \leq M,$$

$$\delta_t V^n = \frac{V^n - V^{n-1}}{\Delta t}, \quad V^{n-\frac{1}{2}} = \frac{1}{2}(V^n + V^{n-1}), \quad 1 \leq n \leq M.$$

Then, the time fractional derivative  ${}_0^C D_t^\alpha u(x, y, t)$  at  $t_{n-\frac{1}{2}}$  can be written

$$(3.1) \quad \begin{aligned} {}_0^C D_t^\alpha u(x, y, t_{n-\frac{1}{2}}) &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_{n-\frac{1}{2}}} \frac{\partial^2 u(x, y, s)}{\partial s^2} \frac{ds}{(t_{n-\frac{1}{2}} - s)^{\alpha-1}} \\ &= \mathcal{J}_\alpha^{n-\frac{1}{2}}(u) + R_\alpha^{n-\frac{1}{2}}, \quad 1 \leq n \leq M, \end{aligned}$$

where

$$(3.2) \quad \mathcal{J}_\alpha^{n-\frac{1}{2}}(u) = \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \left[ b_0 \delta_t u^n - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \delta_t u^j - b_{n-1} \phi \right],$$

with  $b_j = (j+1)^{2-\alpha} - j^{2-\alpha}$ ,  $j \geq 0$ ,  $\phi(x, y) = D_t u(x, y, 0)$  from (1.2). The quantity  $\mathcal{J}_\alpha^{n-\frac{1}{2}}(u)$  is the L1-approximation of the Caputo derivative at  $t_{n-\frac{1}{2}}$ , with truncation error,  $R_\alpha^{n-\frac{1}{2}}$ , satisfying

$$(3.3) \quad \left| R_\alpha^{n-\frac{1}{2}} \right| \leq C \Delta t^{3-\alpha}, \quad 1 \leq n \leq M;$$

see [13, 28]. The coefficients  $b_j$  possess the following properties which are required in subsequent analyses.

LEMMA 3.1. [28] *The coefficients  $b_j$ ,  $j \geq 0$ , satisfy:*

- (i)  $1 = b_0 > b_1 > \cdots > b_n > b_{n+1} > \cdots \rightarrow 0$ ;
- (ii)  $(2 - \alpha)(j + 1)^{1-\alpha} < b_j < (2 - \alpha)j^{1-\alpha}$ ,  $j \geq 1$ ;
- (iii)  $\sum_{j=0}^n (b_j - b_{j+1}) + b_{n+1} = 1$ ;
- (iv)  $\sum_{j=1}^n b_{j-1} = n^{2-\alpha}$ .

With the approximation of the Caputo derivative given by (3.2), the Crank-Nicolson OSC scheme for the approximation of (1.1) consists in find  $U_h^n \in \mathcal{M}(\delta)$ ,  $n = 1, 2, \dots, M$ , such that, for  $1 \leq n \leq M$ ,

$$(3.4) \quad \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} [b_0 \delta_t U_h^n - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \delta_t U_h^j - b_{n-1} \phi] = \Delta U_h^{n-\frac{1}{2}} + f^{n-\frac{1}{2}} \quad \text{on } \Lambda,$$

where  $f^{n-\frac{1}{2}} = f(\cdot, \cdot, t_{n-\frac{1}{2}})$ ;  $U_h^0$  is prescribed later.

With  $E_h^n = U_h^n - U_h^{n-1}$ , we write (3.4) in the form

$$(3.5) \quad \begin{aligned} & \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} [E_h^n - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) E_h^j - \Delta t b_{n-1} \phi] \\ &= \frac{1}{2} \Delta E_h^n + \Delta U_h^{n-1} + f^{n-\frac{1}{2}} \quad \text{on } \Lambda, \quad 1 \leq n \leq M, \end{aligned}$$

since  $b_0 = 1$  from Lemma 3.1(i). Let

$$(3.6) \quad \mu = \Gamma(3-\alpha) \Delta t^\alpha.$$

On multiplying (3.5) by  $\mu$  and rearranging terms, we obtain

$$(3.7) \quad E_h^n - \frac{\mu}{2} \Delta E_h^n = \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) E_h^j + \Delta t b_{n-1} \phi + \mu \Delta U_h^{n-1} + \mu f^{n-\frac{1}{2}} \quad \text{on } \Lambda, \quad 1 \leq n \leq M.$$

On adding the term

$$\frac{\mu^2}{4} \frac{\partial^4 E_h^n}{\partial x^2 \partial y^2}$$

to the left-hand side of (3.7), we obtain:

$$(3.8) \quad \left[ 1 - \frac{\mu}{2} \Delta + \frac{\mu^2}{4} \frac{\partial^4}{\partial x^2 \partial y^2} \right] E_h^n = F^n \quad \text{on } \Lambda, \quad 1 \leq n \leq M,$$

where

$$F^n = \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) E_h^j + \Delta t b_{n-1} \phi + \mu \Delta U_h^{n-1} + \mu f^{n-\frac{1}{2}} + \frac{\mu^2}{4} \frac{\partial^4 E_h^n}{\partial x^2 \partial y^2},$$

the basis of the ADI OSC Crank-Nicolson method for approximating (1.1).

To write (3.8) as an ADI method in matrix-vector form, let  $\{\chi_i\}_{i=1}^{M_x}$  and  $\{\psi_j\}_{j=1}^{M_y}$  be bases for the subspaces  $\mathcal{M}(r, \delta_x)$  and  $\mathcal{M}(r, \delta_y)$ , respectively, where  $M_x = (r-1)N_x$  and  $M_y = (r-1)N_y$ , and set

$$U_h^n(x, y) = \sum_{i=1}^{M_x} \sum_{j=1}^{M_y} \gamma_{ij}^{(n)} \chi_i(x) \psi_j(y).$$

We let

$$\mathbf{\Upsilon}^{(n)} = \left[ \gamma_{11}^{(n)}, \gamma_{12}^{(n)}, \dots, \gamma_{1M_y}^{(n)}, \gamma_{21}^{(n)}, \gamma_{22}^{(n)}, \dots, \gamma_{2M_y}^{(n)}, \gamma_{31}^{(n)}, \dots, \gamma_{M_x M_y}^{(n)} \right]^T,$$

$$\mathbf{F}^{(n)} = [F^n(\xi_1^x, \xi_1^y), F^n(\xi_1^x, \xi_2^y), \dots, F^n(\xi_1^x, \xi_{M_y}^y), F^n(\xi_2^x, \xi_1^y), \dots, F^n(\xi_{M_x}^x, \xi_{M_y}^y)]^T$$

and define the matrices

$$A_x = [-\chi_j''(\xi_i^x)]_{i,j=1}^{M_x}, \quad A_y = [-\psi_j''(\xi_i^y)]_{i,j=1}^{M_y},$$

$$B_x = [\chi_j(\xi_i^x)]_{i,j=1}^{M_x}, \quad B_y = [\psi_j(\xi_i^y)]_{i,j=1}^{M_y}.$$

Then the algebraic problem comprises determining  $\boldsymbol{\nu}^{(n)} = \mathbf{\Upsilon}^{(n)} - \mathbf{\Upsilon}^{(n-1)}$  from

$$(3.9) \quad \left[ \left( B_x + \frac{\mu}{2} A_x \right) \otimes I_{M_y} \right] \hat{\boldsymbol{\nu}}^{(n)} = \mathbf{F}^{(n)},$$

and

$$(3.10) \quad \left[ I_{M_x} \otimes \left( B_y + \frac{\mu}{2} A_y \right) \right] \boldsymbol{\nu}^{(n)} = \hat{\boldsymbol{\nu}}^{(n)},$$

where  $\otimes$  denotes the matrix tensor product, and  $\hat{\boldsymbol{\nu}}^{(n)}$  is an auxiliary vector, cf., [19]. Thus, it follows on using properties of  $\otimes$  that  $\boldsymbol{\nu}^{(n)}$  is determined by solving the two sets of independent one-dimensional problems, (3.9) and (3.10). With standard choices of bases for the spaces  $\mathcal{M}(r, \delta_x)$  and  $\mathcal{M}(r, \delta_y)$ , these linear systems have an almost block diagonal structure, and can be solved efficiently using algorithms described in [9], for example. Clearly, the computation of  $\boldsymbol{\nu}^{(n)}$  is highly parallel.

**4. Stability analysis.** In this section, we derive stability results in the  $H^l$ -norm,  $l = 0, 1, 2$ .

**4.1. The  $H^1$  stability analysis.** An  $H^1$  stability result for (3.8) is proved in the following theorem.

**THEOREM 4.1.** *The ADI OSC Crank-Nicolson method (3.8) is stable with respect to the  $H^1$  norm. Specifically, for  $U_h^n \in \mathcal{M}(\delta)$ ,  $1 \leq n \leq M$ ,*

$$(4.1) \quad \|\nabla U_h^n\|^2 \leq C \|\nabla U_h^0\|^2 + \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)} \|\phi\|_D^2 + \Delta t \sum_{j=1}^n \Gamma(2-\alpha) t_n^{\alpha-1} \left\| f^{j-\frac{1}{2}} \right\|_D^2.$$

*Proof.* First note that (3.8) can be written as

$$(4.2) \quad \begin{aligned} & \delta_t U_h^n - \frac{\mu}{\Delta t} \Delta U_h^{n-\frac{1}{2}} + \frac{\mu^2}{4} \frac{\partial^4 \delta_t U_h^n}{\partial x^2 \partial y^2} \\ &= \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \delta_t U_h^j + b_{n-1} \phi + \frac{\mu}{\Delta t} f^{n-\frac{1}{2}} \quad \text{on } \Lambda, \quad 1 \leq n \leq M, \end{aligned}$$

or, on substituting (3.6) into (4.2) and rearranging terms,

$$\begin{aligned}
 (4.3) \quad & \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \delta_t U_h^n - \Delta U_h^{n-\frac{1}{2}} + \frac{\Gamma(3-\alpha) \Delta t^{1+\alpha}}{4} \frac{\partial^4 \delta_t U_h^n}{\partial x^2 \partial y^2} \\
 &= \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \delta_t U_h^j + \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \phi + f^{n-\frac{1}{2}} \\
 & \quad \text{on } \Lambda, \quad 1 \leq n \leq M.
 \end{aligned}$$

Taking the discrete inner product of (4.3) with  $\delta_t U_h^n$  yields

$$\begin{aligned}
 (4.4) \quad & \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \langle \delta_t U_h^n, \delta_t U_h^n \rangle - \left\langle \Delta U_h^{n-\frac{1}{2}}, \delta_t U_h^n \right\rangle \\
 & + \frac{\Gamma(3-\alpha) \Delta t^{1+\alpha}}{4} \left\langle \frac{\partial^4 \delta_t U_h^n}{\partial x^2 \partial y^2}, \delta_t U_h^n \right\rangle \\
 &= \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \left\langle \delta_t U_h^j, \delta_t U_h^n \right\rangle \\
 & + \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \langle \phi, \delta_t U_h^n \rangle + \left\langle f^{n-\frac{1}{2}}, \delta_t U_h^n \right\rangle, \quad 1 \leq n \leq M.
 \end{aligned}$$

The first term on the left-hand side of (4.4) can be written as

$$(4.5) \quad \langle \delta_t U_h^n, \delta_t U_h^n \rangle = \|\delta_t U_h^n\|_D^2.$$

A straightforward calculation shows that the second term on the left-hand side of (4.4) gives

$$(4.6) \quad - \left\langle \Delta U_h^{n-\frac{1}{2}}, \delta_t U_h^n \right\rangle = \frac{1}{2} \delta_t \langle -\Delta U_h^n, U_h^n \rangle.$$

and, from Lemma 2.2, we have, for the third term,

$$(4.7) \quad \left\langle \frac{\partial^4 [\delta_t U_h^n]}{\partial x^2 \partial y^2}, \delta_t U_h^n \right\rangle \geq \left\| \frac{\partial^2 \delta_t U_h^n}{\partial x \partial y} \right\|^2 \geq 0.$$

On substituting (4.5) and (4.6) into (4.4) and dropping the non-negative term,  $\left\| \frac{\partial^2 \delta_t U_h^n}{\partial x \partial y} \right\|^2$ , we obtain

$$\begin{aligned}
 (4.8) \quad & \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \|\delta_t U_h^n\|_D^2 + \frac{1}{2} \delta_t \langle -\Delta U_h^n, U_h^n \rangle \\
 & \leq \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \left\langle \delta_t U_h^j, \delta_t U_h^n \right\rangle \\
 & + \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \langle \phi, \delta_t U_h^n \rangle + \left\langle f^{n-\frac{1}{2}}, \delta_t U_h^n \right\rangle, \quad 1 \leq n \leq M.
 \end{aligned}$$

Multiplying (4.8) by  $2\Delta t$ , and using the fact that, from Lemma 3.1(i),  $b_{n-1} > 0$  and  $b_{n-j-1} - b_{n-j} > 0$ , we obtain, for  $1 \leq n \leq M$ ,

$$(4.9) \quad \frac{2\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \|\delta_t U_h^n\|_D^2 + \langle -\Delta U_h^n, U_h^n \rangle$$



$$\begin{aligned}
&\leq \langle -\Delta U_h^{n-1}, U_h^{n-1} \rangle + \frac{2\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \langle \delta_t U_h^j, \delta_t U_h^n \rangle \\
&\quad + \frac{2\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \langle \phi, \delta_t U_h^n \rangle + 2\Delta t \langle f^{n-\frac{1}{2}}, \delta_t U_h^n \rangle.
\end{aligned}$$

On using the Cauchy-Schwarz inequality and the triangle inequality, (4.9) can be rewritten as

$$\begin{aligned}
(4.10) \quad &\frac{2\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \|\delta_t U_h^n\|_D^2 + \langle -\Delta U_h^n, U_h^n \rangle \\
&\leq \langle -\Delta U_h^{n-1}, U_h^{n-1} \rangle + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \left[ \|\delta_t U_h^j\|_D^2 + \|\delta_t U_h^n\|_D^2 \right] \\
&\quad + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \left[ \|\phi\|_D^2 + \|\delta_t U_h^n\|_D^2 \right] + 2\Delta t \langle f^{n-\frac{1}{2}}, \delta_t U_h^n \rangle \\
&= \langle -\Delta U_h^{n-1}, U_h^{n-1} \rangle + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \|\delta_t U_h^j\|_D^2 \\
&\quad + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \left[ \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) + b_{n-1} \right] \|\delta_t U_h^n\|_D^2 \\
&\quad + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \|\phi\|_D^2 + 2\Delta t \langle f^{n-\frac{1}{2}}, \delta_t U_h^n \rangle, \quad 1 \leq n \leq M.
\end{aligned}$$

Note that, from Lemma 3.1(i) and (iii),

$$\sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) + b_{n-1} = b_0 = 1,$$

so that (4.10) becomes

$$\begin{aligned}
(4.11) \quad &\frac{2\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \|\delta_t U_h^n\|_D^2 + \langle -\Delta U_h^n, U_h^n \rangle \\
&\leq \langle -\Delta U_h^{n-1}, U_h^{n-1} \rangle + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \|\delta_t U_h^j\|_D^2 \\
&\quad + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \|\delta_t U_h^n\|_D^2 + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \|\phi\|_D^2 \\
&\quad + 2\Delta t \langle f^{n-\frac{1}{2}}, \delta_t U_h^n \rangle, \quad 1 \leq n \leq M.
\end{aligned}$$

On reformulating (4.11), we obtain

$$\begin{aligned}
(4.12) \quad &\langle -\Delta U_h^n, U_h^n \rangle + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^n b_{n-j} \|\delta_t U_h^j\|_D^2 \\
&\leq \langle -\Delta U_h^{n-1}, U_h^{n-1} \rangle + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} b_{n-j-1} \|\delta_t U_h^j\|_D^2 \\
&\quad + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \|\phi\|_D^2 + 2\Delta t \langle f^{n-\frac{1}{2}}, \delta_t U_h^n \rangle, \quad 1 \leq n \leq M.
\end{aligned}$$

For convenience, we define  $G^n$  by:

$$(4.13) \quad \begin{cases} G^0 = \langle -\Delta U_h^0, U_h^0 \rangle, \\ G^n = \langle -\Delta U_h^n, U_h^n \rangle + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^n b_{n-j} \left\| \delta_t U_h^j \right\|_D^2, \quad n \geq 1. \end{cases}$$

Then (4.12) is equivalent to

$$(4.14) \quad \begin{aligned} G^n &\leq G^{n-1} + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \|\phi\|_D^2 + 2\Delta t \left\langle f^{n-\frac{1}{2}}, \delta_t U_h^n \right\rangle \\ &\leq G^{n-2} + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} b_{n-2} \|\phi\|_D^2 + 2\Delta t \left\langle f^{n-1-\frac{1}{2}}, \delta_t U_h^{n-1} \right\rangle \\ &\quad + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \|\phi\|_D^2 + 2\Delta t \left\langle f^{n-\frac{1}{2}}, \delta_t U_h^n \right\rangle \\ &\quad \dots\dots\dots \\ &\leq G^0 + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \left[ \sum_{j=1}^n b_{n-j} \right] \|\phi\|_D^2 + 2\Delta t \sum_{j=1}^n \left\langle f^{j-\frac{1}{2}}, \delta_t U_h^j \right\rangle. \\ &\leq G^0 + \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)} \|\phi\|_D^2 + 2\Delta t \sum_{j=1}^n \left\langle f^{j-\frac{1}{2}}, \delta_t U_h^j \right\rangle, \end{aligned}$$

using Lemma 3.1(iv) in the last step. On using the Cauchy-Schwarz inequality and the Young's inequality (2.2), the last term on the right hand side of (4.14) may be bounded as

$$(4.15) \quad \begin{aligned} &2\Delta t \sum_{j=1}^n \left\langle f^{j-\frac{1}{2}}, \delta_t U_h^j \right\rangle \\ &\leq \Delta t \sum_{j=1}^n \left[ \frac{\Gamma(3-\alpha)}{\Delta t^{1-\alpha} b_{n-j}} \left\| f^{j-\frac{1}{2}} \right\|_D^2 + \frac{\Delta t^{1-\alpha} b_{n-j}}{\Gamma(3-\alpha)} \left\| \delta_t U_h^j \right\|_D^2 \right]. \end{aligned}$$

On substituting (4.13) and (4.15) into (4.14) and simplifying the resulting expression, we obtain, for  $1 \leq n \leq M$ ,

$$(4.16) \quad \begin{aligned} \langle -\Delta U_h^n, U_h^n \rangle &\leq \langle -\Delta U_h^0, U_h^0 \rangle + \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)} \|\phi\|_D^2 \\ &\quad + \Delta t \sum_{j=1}^n \frac{\Delta t^{\alpha-1} \Gamma(3-\alpha)}{b_{n-j}} \left\| f^{j-\frac{1}{2}} \right\|_D^2, \quad 1 \leq n \leq M. \end{aligned}$$

Also, since, from Lemma 3.1(ii),

$$b_{n-j} \geq (2-\alpha)(n-j+1)^{1-\alpha} \geq (2-\alpha)n^{1-\alpha},$$

the last term on the right hand side of (4.16) can be bounded as

$$(4.17) \quad \begin{aligned} \Delta t \sum_{j=1}^n \frac{\Delta t^{\alpha-1} \Gamma(3-\alpha)}{b_{n-j}} \left\| f^{j-\frac{1}{2}} \right\|_D^2 &\leq \Delta t \sum_{j=1}^n \frac{\Delta t^{\alpha-1} \Gamma(3-\alpha)}{(2-\alpha)n^{1-\alpha}} \left\| f^{j-\frac{1}{2}} \right\|_D^2 \\ &\leq \Delta t \sum_{j=1}^n \Gamma(2-\alpha) t_n^{\alpha-1} \left\| f^{j-\frac{1}{2}} \right\|_D^2. \end{aligned}$$

Using (2.4) and (2.5), it can be shown that there exist positive constants  $C_1$  and  $C_2$  such that

$$(4.18) \quad \langle -\Delta U_h^n, U_h^n \rangle \geq C_1 \|\nabla U_h^n\|^2, \quad \langle -\Delta U_h^0, U_h^0 \rangle \leq C_2 \|\nabla U_h^0\|^2.$$

Substituting (4.17) and (4.18) into (4.16), and rearranging, we obtain

$$(4.19) \quad \|\nabla U_h^n\|^2 \leq C \|\nabla U_h^0\|^2 + \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)} \|\phi\|_D^2 + \Delta t \sum_{j=1}^n \Gamma(2-\alpha) t_n^{\alpha-1} \|f^{j-\frac{1}{2}}\|_D^2,$$

which completes the proof.  $\square$

**4.2. The  $H^2$  stability analysis.** An  $H^2$  stability estimate is derived in the following theorem.

**THEOREM 4.2.** *The ADI OSC Crank-Nicolson method (3.8) is stable with respect to the  $H^2$  norm. More precisely, for  $U_h^n \in \mathcal{M}(\delta)$ ,  $1 \leq n, q \leq M$ , we have*

$$\|U_h^n\|_{H^2}^2 \leq C \left[ \|\Delta U_h^0\|_D^2 + t_q^{2-2\alpha} \|\phi\|_D^2 + \|f^{\frac{1}{2}}\|_D^2 + \|f^{n-\frac{1}{2}}\|_D^2 + \Delta t \sum_{j=2}^n \|\delta_t f^{j-\frac{1}{2}}\|_D^2 \right].$$

*Proof.* Taking the inner product of (4.3) with  $-\Delta \delta_t U_h^n$ , we obtain

$$(4.20) \quad \begin{aligned} & \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \langle \delta_t U_h^n, -\Delta \delta_t U_h^n \rangle - \left\langle \Delta U_h^{n-\frac{1}{2}}, -\Delta \delta_t U_h^n \right\rangle \\ & + \frac{\Gamma(3-\alpha) \Delta t^{1+\alpha}}{4} \left\langle \frac{\partial^4 \delta_t U_h^n}{\partial x^2 \partial y^2}, -\Delta \delta_t U_h^n \right\rangle \\ & = \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \left\langle \delta_t U_h^j, -\Delta \delta_t U_h^n \right\rangle \\ & + \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \langle \phi, -\Delta \delta_t U_h^n \rangle + \left\langle f^{n-\frac{1}{2}}, -\Delta \delta_t U_h^n \right\rangle, \quad 1 \leq n \leq M. \end{aligned}$$

On using (2.4) with  $U = \delta_t U_h^n$ , we have, in the first term on the right hand side,

$$(4.21) \quad \langle \delta_t U_h^n, -\Delta \delta_t U_h^n \rangle \geq C \|\nabla \delta_t U_h^n\|^2.$$

The second term on the left-hand side of (4.20) can be written as

$$(4.22) \quad \left\langle \Delta U_h^{n-\frac{1}{2}}, \Delta \delta_t U_h^n \right\rangle = \frac{1}{2} \delta_t \langle \Delta U_h^n, \Delta U_h^n \rangle = \frac{1}{2} \delta_t \|\Delta U_h^n\|_D^2,$$

From (2.7) with  $V = \delta_t U_h^n$ ,

$$(4.23) \quad \left\langle \frac{\partial^4 \delta_t U_h^n}{\partial x^2 \partial y^2}, -\Delta \delta_t U_h^n \right\rangle \geq \left\| \frac{\partial^3 \delta_t U_h^n}{\partial x^2 \partial y} \right\|^2 + \left\| \frac{\partial^3 \delta_t U_h^n}{\partial x \partial y^2} \right\|^2 \geq 0.$$

On substituting (4.21)–(4.23) into (4.20), dropping the non-negative terms  $\left\| \frac{\partial^3 \delta_t U_h^n}{\partial x^2 \partial y} \right\|^2$

and  $\left\| \frac{\partial^3 \delta_t U_h^n}{\partial x \partial y^2} \right\|^2$ , we obtain

$$(4.24) \quad \frac{C_1 \Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \|\nabla \delta_t U_h^n\|^2 + \frac{1}{2} \delta_t \|\Delta U_h^n\|_D^2$$

$$\begin{aligned}
&\leq \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \left\langle \delta_t U_h^j, -\Delta \delta_t U_h^n \right\rangle \\
&\quad + \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \left\langle \phi, -\Delta \delta_t U_h^n \right\rangle + \left\langle f^{n-\frac{1}{2}}, -\Delta \delta_t U_h^n \right\rangle, \quad 1 \leq n \leq M.
\end{aligned}$$

Since  $b_{n-j-1} - b_{n-j} > 0$  from Lemma 3.1(i),

$$\begin{aligned}
(4.25) \quad &\sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \left\langle \delta_t U_h^j, -\Delta \delta_t U_h^n \right\rangle \\
&\leq \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \left| \left\langle \delta_t U_h^j, -\Delta \delta_t U_h^n \right\rangle \right| \\
&\leq \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \left[ \frac{1}{\varepsilon} \left\| \nabla \delta_t U_h^j \right\|^2 + \varepsilon \left\| \nabla \delta_t U_h^n \right\|^2 \right] \\
&\leq \frac{1}{\varepsilon} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \left\| \nabla \delta_t U_h^j \right\|^2 + \varepsilon \left\| \nabla \delta_t U_h^n \right\|^2,
\end{aligned}$$

on using (2.5), (2.2) and the fact that  $\sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) = 1 - b_{n-1} < 1$  from Lemma 3.1(iii). We multiply (4.24) by  $2\Delta t$ , use (4.25) and rearrange terms to obtain

$$\begin{aligned}
(4.26) \quad &\left\| \Delta U_h^n \right\|_D^2 + \frac{C_3 \Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^n b_{n-j} \left\| \nabla \delta_t U_h^j \right\|^2 \\
&\leq \left\| \Delta U_h^{n-1} \right\|_D^2 + \frac{C_4 \Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} b_{n-j-1} \left\| \nabla \delta_t U_h^j \right\|^2 \\
&\quad + \frac{2\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \left\langle \phi, -\Delta \delta_t U_h^n \right\rangle + 2\Delta t \left\langle f^{n-\frac{1}{2}}, -\Delta \delta_t U_h^n \right\rangle, \quad 1 \leq n \leq M.
\end{aligned}$$

Therefore, using arguments similar to those in (4.14), we have, for  $1 \leq n \leq M$ ,

$$\begin{aligned}
(4.27) \quad &\left\| \Delta U_h^n \right\|_D^2 + \frac{C_3 \Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^n b_{n-j} \left\| \nabla \delta_t U_h^j \right\|^2 \\
&\leq \left\| \Delta U_h^0 \right\|_D^2 + \frac{2\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^n b_{j-1} \left\langle \phi, -\Delta \delta_t U_h^j \right\rangle + 2\Delta t \sum_{j=1}^n \left\langle f^{j-\frac{1}{2}}, -\Delta \delta_t U_h^j \right\rangle.
\end{aligned}$$

In the Appendix A, it is shown that there exist integers  $m$  and  $q$ , with  $1 \leq m \leq n$ ,  $0 \leq q \leq n-1$ , such that

$$\begin{aligned}
(4.28) \quad &\Delta t \sum_{j=1}^n b_{j-1} \left\langle \phi, -\Delta \delta_t U_h^j \right\rangle = \sum_{j=1}^n b_{j-1} \left\langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \right\rangle \\
&\leq b_q \left\langle \phi, \Delta U_h^m - \Delta U_h^0 \right\rangle,
\end{aligned}$$

since  $b_0 = 1$ ,  $b_q \leq (2-\alpha)q^{1-\alpha}$ ,  $q > 0$ , and  $t_1 = \Delta t$ . Using (2.2) and (4.28), the Cauchy-Schwarz and triangle inequality, for  $1 \leq q \leq n-1$ ,  $1 \leq m \leq n$ , we then obtain

$$(4.29) \quad \frac{2\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^n b_{j-1} \left\langle \phi, -\Delta \delta_t U_h^j \right\rangle \leq \frac{2}{\Gamma(3-\alpha)} \left\langle t_q^{1-\alpha} \phi, \Delta U_h^m - \Delta U_h^0 \right\rangle$$

$$\leq \frac{1}{\Gamma(3-\alpha)} \left[ \frac{t_q^{2-2\alpha}}{\varepsilon} \|\phi\|_D^2 + \varepsilon \|\Delta U_h^m\|_D^2 + \varepsilon \|\Delta U_h^0\|_D^2 \right].$$

The last term on the right-hand side of (4.27) is bounded as in [19, Eq. (2.35)] to obtain

$$(4.30) 2\Delta t \left| \sum_{j=1}^n \left\langle f^{j-\frac{1}{2}}, -\Delta \delta_t U_h^j \right\rangle \right| \leq \frac{1}{\varepsilon} \|f^{n-\frac{1}{2}}\|_D^2 + \varepsilon \|\Delta U_h^n\|_D^2 + \frac{1}{\varepsilon} \|f^{\frac{1}{2}}\|_D^2 + \varepsilon \|\Delta U_h^0\|_D^2 \\ + \Delta t \sum_{j=2}^n \|\delta_t f^{j-\frac{1}{2}}\|_D^2 + \Delta t \sum_{j=0}^{n-1} \|\Delta U_h^j\|_D^2.$$

On substituting (4.29) and (4.30) into (4.27), dropping the non-negative second term on the left-hand side of (4.27), and simplifying the resulting expression, we obtain, for  $1 \leq m \leq n$ ,  $1 \leq q \leq n-1$ ,  $1 \leq n \leq M$ ,

$$(4.31) \quad \|\Delta U_h^n\|_D^2 \leq C \left[ \|\Delta U_h^0\|_D^2 + t_q^{2-2\alpha} \|\phi\|_D^2 + \|f^{\frac{1}{2}}\|_D^2 + \|f^{n-\frac{1}{2}}\|_D^2 \right. \\ \left. + \Delta t \sum_{j=2}^n \|\delta_t f^{j-\frac{1}{2}}\|_D^2 \right] + C\Delta t \sum_{j=1}^{n-1} \|\Delta U_h^j\|_D^2 + C\varepsilon \|\Delta U_h^m\|_D^2.$$

Suppose

$$\max_{0 \leq \ell \leq n} \|\Delta U_h^\ell\|_D = \|\Delta U_h^J\|_D.$$

Then

$$\|\Delta U_h^n\|_D^2 \leq C \left[ \|\Delta U_h^0\|_D^2 + t_q^{2-2\alpha} \|\phi\|_D^2 + \|f^{\frac{1}{2}}\|_D^2 + \|f^{n-\frac{1}{2}}\|_D^2 \right. \\ \left. + \Delta t \sum_{j=2}^n \|\delta_t f^{j-\frac{1}{2}}\|_D^2 \right] + C\Delta t \sum_{j=1}^{n-1} \|\Delta U_h^j\|_D^2 + C\varepsilon \|\Delta U_h^J\|_D^2.$$

Thus, since  $0 \leq J \leq n$ ,

$$\|\Delta U_h^J\|_D^2 \leq C \left[ \|\Delta U_h^0\|_D^2 + t_q^{2-2\alpha} \|\phi\|_D^2 + \|f^{\frac{1}{2}}\|_D^2 + \|f^{n-\frac{1}{2}}\|_D^2 \right. \\ \left. + \Delta t \sum_{j=2}^n \|\delta_t f^{j-\frac{1}{2}}\|_D^2 \right] + C\Delta t \sum_{j=1}^{n-1} \|\Delta U_h^j\|_D^2 + C\varepsilon \|\Delta U_h^J\|_D^2,$$

from which it follows that, for  $\varepsilon$  sufficiently small,

$$(4.32) \quad \|\Delta U_h^n\|_D^2 \leq \|\Delta U_h^J\|_D^2 \leq C \left[ \|\Delta U_h^0\|_D^2 + t_q^{2-2\alpha} \|\phi\|_D^2 + \|f^{\frac{1}{2}}\|_D^2 + \|f^{n-\frac{1}{2}}\|_D^2 \right. \\ \left. + \Delta t \sum_{j=2}^n \|\delta_t f^{j-\frac{1}{2}}\|_D^2 \right] + C\Delta t \sum_{j=1}^{n-1} \|\Delta U_h^j\|_D^2.$$

Thus, on applying the discrete Gronwall lemma,

$$\|\Delta U_h^n\|_D^2 \leq C \left[ \|\Delta U_h^0\|_D^2 + t_q^{2-2\alpha} \|\phi\|_D^2 + \|f^{\frac{1}{2}}\|_D^2 + \|f^{n-\frac{1}{2}}\|_D^2 + \Delta t \sum_{j=2}^n \|\delta_t f^{j-\frac{1}{2}}\|_D^2 \right].$$

□

**5. Convergence analysis.** In this section, we give an analysis of the convergence of the ADI OSC method. For this purpose, we introduce the elliptic projection  $W : [0, T] \rightarrow \mathcal{M}(\delta)$  defined by

$$(5.1) \quad \Delta(u - W) = 0 \quad \text{on } \Lambda \times [0, T],$$

where  $u$  is the solution of (1.1)–(1.3). The following two lemmas provide estimates for  $u - W$  and its time derivatives; see [19, Eqs. (2.45), (2.46)].

LEMMA 5.1. *If  $\partial^i u / \partial t^l \in H^{r+3-j}$ ,  $i = 0, 1, 2$ ,  $j = 0, 1, 2$ , and  $W$  is defined by (5.1), then*

$$(5.2) \quad \left\| \frac{\partial^l(u - W)}{\partial t^l} \right\|_{H^j} \leq C h^{r+1-j} \left\| \frac{\partial^l u}{\partial t^l} \right\|_{H^{r+3-j}}, \quad j = 0, 1, 2, \quad l = 0, 1, 2.$$

LEMMA 5.2. *If  $\partial^i u / \partial t^i \in H^{r+3}$ , for  $t \in [0, T]$ ,  $i = 0, 1, 2$ , then*

$$(5.3) \quad \left\| \frac{\partial^{l+i}(u - W)}{\partial x^{l_1} \partial y^{l_2} \partial t^i} \right\|_D \leq C h^{r+1-l} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{H^{r+3}}, \quad 0 \leq l = l_1 + l_2 \leq 4.$$

Convergence results for the ADI OSC method are given in the following theorem.

THEOREM 5.3. *Suppose  $u$  is the solution of (1.1)–(1.3), and  $U_h^n$ ,  $n = 1, 2, \dots, M$ , satisfies (3.8) with  $U_h^0 = W^0$ . If  $u \in C^{2,0,3} \cap C^{0,2,3} \cap C^{2,3,1} \cap C^{3,2,1} \cap C^{0,0,4}$  and  $\partial u / \partial t$ ,  $\partial^2 u / \partial t^2$ ,  $\partial^3 u / \partial t^3 \in C([0, T], H^{r+3})$ , then*

$$(5.4) \quad \|u(t_n) - U_h^n\|_{H^j} \leq C (h^{r+1-j} + \Delta t^{3-\alpha}), \quad j = 0, 1, 2.$$

*Proof.* With  $W$  defined in (5.1), we set

$$(5.5) \quad \eta^n = u^n - W^n, \quad \zeta^n = U_h^n - W^n, \quad 0 \leq n \leq M,$$

so that

$$(5.6) \quad u^n - U_h^n = \eta^n - \zeta^n.$$

Since estimates of  $\eta^n$  are known from Lemmas 5.1 and 5.2, it is sufficient to bound  $\zeta^n$ , then use the triangle inequality to bound  $u^n - U_h^n$ .

From (1.1) and (3.1), it follows that, for  $1 \leq n \leq M$ ,

$$(5.7) \quad \begin{aligned} & \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} [b_0 \delta_t u^n - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \delta_t u^j - b_{n-1} \phi] \\ & + R_\alpha^{n-\frac{1}{2}} + \frac{\Gamma(3-\alpha) \Delta t^{1+\alpha}}{4} \frac{\partial^4 \delta_t u^n}{\partial x^2 \partial y^2} \\ & = \Delta u^{n-\frac{1}{2}} + \frac{\Gamma(3-\alpha) \Delta t^{1+\alpha}}{4} \frac{\partial^4 \delta_t u^n}{\partial x^2 \partial y^2} + f(t_{n-\frac{1}{2}}), \quad \text{on } \Lambda. \end{aligned}$$

Combining (5.7), (4.3), (3.8), (5.1) and (5.5), we have

$$(5.8) \quad \begin{aligned} & \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \delta_t \zeta^n - \Delta \zeta^{n-\frac{1}{2}} + \frac{\Gamma(3-\alpha) \Delta t^{1+\alpha}}{4} \frac{\partial^4 \delta_t \zeta^n}{\partial x^2 \partial y^2} \\ &= \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \delta_t \zeta^j + \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} b_{n-1} \delta_t \eta^1 + \mathcal{F}_u^{n-\frac{1}{2}}, \end{aligned}$$

where

$$(5.9) \quad \begin{aligned} \mathcal{F}_u^{n-\frac{1}{2}} &= R_\alpha^{n-\frac{1}{2}} - \frac{\Gamma(3-\alpha) \Delta t^{1+\alpha}}{4} \frac{\partial^4 \delta_t u^n}{\partial x^2 \partial y^2} + \frac{\Gamma(3-\alpha) \Delta t^{1+\alpha}}{4} \frac{\partial^4 \delta_t \eta^n}{\partial x^2 \partial y^2} \\ &\quad + \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} b_{n-j-1} (\delta_t \eta^{j+1} - \delta_t \eta^j). \end{aligned}$$

We first prove (5.4) for  $j = 0, 1$ . Applying the stability result (4.1) of Theorem 4.1 to (5.8), we obtain

$$(5.10) \quad \|\nabla \zeta^n\|^2 \leq C \|\nabla \zeta^0\|^2 + \frac{t_n^{2-\alpha} \|\delta_t \eta^1\|_D^2}{\Gamma(3-\alpha)} + \Delta t \sum_{j=1}^n \Gamma(2-\alpha) t_n^{\alpha-1} \|\mathcal{F}_u^{j-\frac{1}{2}}\|_D^2$$

From (5.9), we have

$$(5.11) \quad \begin{aligned} \|\mathcal{F}_u^{n-\frac{1}{2}}\|_D &\leq \|R_\alpha^{n-\frac{1}{2}}\|_D + \frac{\Gamma(3-\alpha) \Delta t^{1+\alpha}}{4} \left\| \frac{\partial^4 \delta_t \eta^n}{\partial x^2 \partial y^2} - \frac{\partial^4 \delta_t u^n}{\partial x^2 \partial y^2} \right\|_D \\ &\quad + \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \left\| \sum_{j=1}^{n-1} b_{n-j-1} (\delta_t \eta^{j+1} - \delta_t \eta^j) \right\|_D. \end{aligned}$$

From (3.3),

$$(5.12) \quad \|R_\alpha^{n-\frac{1}{2}}\|_D \leq C \Delta t^{3-\alpha}, \quad 1 \leq n \leq M.$$

For the second term on the right-hand side of (5.11), we obtain

$$(5.13) \quad \begin{aligned} & \frac{\Gamma(3-\alpha) \Delta t^{1+\alpha}}{4} \left\| \frac{\partial^4 \delta_t \eta^n}{\partial x^2 \partial y^2} - \frac{\partial^4 \delta_t u^n}{\partial x^2 \partial y^2} \right\|_D \\ & \leq C \Delta t^{1+\alpha} \left[ \left\| \frac{\partial u}{\partial t} \right\|_{C^{2,2,0}} + \left\| \frac{\partial u}{\partial t} \right\|_{C([0,T],H^{r+3})} \right], \end{aligned}$$

from [19, Eqs. (2.58), (2.59)].

The last term on the right-hand side in (5.11) is bounded in the following way. First,

$$(5.14) \quad \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \left\| \sum_{j=1}^{n-1} b_{n-j-1} (\delta_t \eta^{j+1} - \delta_t \eta^j) \right\|_D \leq \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-1} b_{n-j-1} \|\delta_t^2 \eta^{j+1}\|_D,$$

and

$$\begin{aligned}
(5.15) \quad \|\delta_t^2 \eta^{j+1}\|_D &= \frac{1}{\Delta t^2} \left\| \int_{t_{j-1}}^{t_j} (\tau - t_{j-1}) \frac{\partial^2 \eta}{\partial t^2}(\tau) d\tau - \int_{t_j}^{t_{j+1}} (\tau - t_{j+1}) \frac{\partial^2 \eta}{\partial t^2}(\tau) d\tau \right\|_D \\
&\leq \frac{1}{\Delta t} \left[ \int_{t_{j-1}}^{t_j} \left\| \frac{\partial^2 \eta}{\partial t^2}(\tau) \right\|_D d\tau + \int_{t_j}^{t_{j+1}} \left\| \frac{\partial^2 \eta}{\partial t^2}(\tau) \right\|_D d\tau \right] \\
&\leq Ch^{r+1} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{C([0,T],H^{r+3})}, \quad j \geq 1,
\end{aligned}$$

using Lemma 5.2. Hence, on substituting (5.15) into (5.14), and using  $\sum_{j=1}^{n-1} b_{n-j-1} = (n-1)^{2-\alpha}$  from Lemma 3.1(iv), we have

$$\begin{aligned}
(5.16) \quad \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \left\| \sum_{j=1}^{n-1} b_{n-j-1} (\delta_t \eta^{j+1} - \delta_t \eta^j) \right\|_D &\leq C_1 h^{r+1} \frac{t_{n-1}^{2-\alpha}}{\Gamma(3-\alpha)} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{C([0,T],H^{r+3})} \\
&\leq CT^{2-\alpha} h^{r+1}.
\end{aligned}$$

Since  $1 < \alpha < 2$ , we have  $1 + \alpha > 3 - \alpha$ . Therefore, with (5.12), (5.13) and (5.16) in (5.11), we obtain

$$(5.17) \quad \left\| \mathcal{F}_u^{n-\frac{1}{2}} \right\|_D \leq C (\Delta t^{3-\alpha} + h^{r+1}).$$

Also,

$$\begin{aligned}
(5.18) \quad \|\delta_t \eta^1\|_D &= \frac{1}{\Delta t} \left\| \int_{t_0}^{t_1} \frac{\partial \eta}{\partial t}(\tau) d\tau \right\|_D \leq \frac{1}{\Delta t} \int_{t_0}^{t_1} \left\| \frac{\partial \eta}{\partial t}(\tau) \right\|_D d\tau \\
&\leq Ch^{r+1} \left\| \frac{\partial u}{\partial t} \right\|_{C([0,T],H^{r+3})},
\end{aligned}$$

using (5.3). Then, substituting (5.17) and (5.18) into (5.10), and noting that  $\zeta^0 = 0$ , we have

$$(5.19) \quad \|\nabla \zeta^n\| \leq C (\Delta t^{3-\alpha} + h^{r+1}).$$

Using Poincaré's inequality, it follows that

$$(5.20) \quad \|\zeta^n\| \leq C (\Delta t^{3-\alpha} + h^{r+1}).$$

Then using the triangle inequality, (5.19), (5.20) and Lemma 5.1, it follows that

$$\|u(t_n) - U_h^n\|_{H^j} \leq C (h^{r+1-j} + \Delta t^{3-\alpha}), \quad j = 0, 1,$$

as desired.

When  $j = 2$ , according to Theorem 4.2, we obtain

$$\begin{aligned}
(5.21) \quad \|\zeta^n\|_{H^2}^2 &\leq C \left( \|\Delta \zeta^0\|_D^2 + t_q^{2-2\alpha} \|\delta_t \eta^1\|_D^2 \right) \\
&\quad + C \left( \left\| \mathcal{F}_u^{\frac{1}{2}} \right\|_D^2 + \left\| \mathcal{F}_u^{n-\frac{1}{2}} \right\|_D^2 + \Delta t \sum_{j=2}^n \left\| \delta_t \mathcal{F}_u^{j-\frac{1}{2}} \right\|_D^2 \right).
\end{aligned}$$



In order to complete the proof, we require an estimate of  $\left\|\delta_t \mathcal{F}_u^{n-\frac{1}{2}}\right\|_D^2$ . First observe that, from (5.9), for  $2 \leq n \leq M$ ,

$$(5.22) \quad \delta_t \mathcal{F}_u^{n-\frac{1}{2}} = \delta_t R_\alpha^{n-\frac{1}{2}} - \frac{\Gamma(3-\alpha)\Delta t^{1+\alpha}}{4} \frac{\partial^4 \delta_t^2 u^n}{\partial x^2 \partial y^2} + \frac{\Gamma(3-\alpha)\Delta t^{1+\alpha}}{4} \frac{\partial^4 \delta_t^2 \eta^n}{\partial x^2 \partial y^2} \\ + \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \left[ \sum_{j=1}^{n-2} b_{j-1} (\delta_t^2 \eta^{n-j+1} - \delta_t^2 \eta^{n-j}) + b_{n-2} \delta_t^2 \eta^2 \right],$$

from which it follows that

$$(5.23) \quad \left\|\delta_t \mathcal{F}_u^{n-\frac{1}{2}}\right\|_D \leq \left\|\delta_t R_\alpha^{n-\frac{1}{2}}\right\|_D + \frac{\Gamma(3-\alpha)\Delta t^{1+\alpha}}{4} \left\| \frac{\partial^4 \delta_t^2 \eta^n}{\partial x^2 \partial y^2} - \frac{\partial^4 \delta_t^2 u^n}{\partial x^2 \partial y^2} \right\|_D \\ + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \left\| \sum_{j=1}^{n-2} b_{j-1} \delta_t^3 \eta^{n-j+1} \right\|_D + \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} b_{n-2} \left\| \delta_t^2 \eta^2 \right\|_D.$$

In Appendix B, it is proved that

$$(5.24) \quad \left\|\delta_t R_\alpha^{n-\frac{1}{2}}\right\|_D \leq C \Delta t^{3-\alpha}, \quad 1 \leq n \leq M.$$

Then

$$(5.25) \quad \frac{\Gamma(3-\alpha)\Delta t^{1+\alpha}}{4} \left\| \frac{\partial^4 \delta_t^2 \eta^n}{\partial x^2 \partial y^2} - \frac{\partial^4 \delta_t^2 u^n}{\partial x^2 \partial y^2} \right\|_D \\ \leq C \Delta t^{1+\alpha} \left[ \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{C^{2,2,0}} + \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{C([0,T],H^{r+3})} \right];$$

cf., (5.13). Also, for  $j \geq 3$ ,

$$(5.26) \quad \left\| \delta_t^3 \eta^j \right\|_D = \frac{1}{\Delta t^3} \left\| \int_{t_{j-1}}^{t_j} \frac{1}{2} (s - t_j)^2 \frac{\partial^3 \eta}{\partial t^3}(\cdot, s) ds - \int_{t_{j-2}}^{t_{j-1}} \frac{1}{2} (s - t_{j-2})^2 \frac{\partial^3 \eta}{\partial t^3}(\cdot, s) ds \right. \\ \left. + \int_{t_{j-3}}^{t_{j-2}} \frac{1}{2} (s - t_{j-3})^2 \frac{\partial^3 \eta}{\partial t^3}(\cdot, s) ds - \int_{t_{j-2}}^{t_{j-1}} \frac{1}{2} (s - t_{j-1})^2 \frac{\partial^3 \eta}{\partial t^3}(\cdot, s) ds \right\|_D \\ \leq \frac{1}{\Delta t} \left[ \frac{1}{2} \int_{t_{j-1}}^{t_j} \left\| \frac{\partial^3 \eta}{\partial t^3} \right\|_D ds + \int_{t_{j-2}}^{t_{j-1}} \left\| \frac{\partial^3 \eta}{\partial t^3} \right\|_D ds + \frac{1}{2} \int_{t_{j-3}}^{t_{j-2}} \left\| \frac{\partial^3 \eta}{\partial t^3} \right\|_D ds \right] \\ \leq C h^{r+1} \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{C([0,T],H^{r+3})},$$

using Lemma 5.2. Thus, since  $\sum_{j=1}^{n-2} b_{j-1} = (n-2)^{2-\alpha}$  from Lemma 3.1(iv), we obtain, on using (5.26),

$$(5.27) \quad \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \left\| \sum_{j=1}^{n-2} b_{j-1} \delta_t^3 \eta^{n-j+1} \right\|_D \leq \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^{n-2} b_{j-1} \left\| \delta_t^3 \eta^{n-j+1} \right\|_D \\ \leq \frac{C h^{r+1} \Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{C([0,T],H^{r+3})},$$

for  $n \geq 3$ , and, from (5.15),

$$(5.28) \quad \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} b_{n-2} \|\delta_t^2 \eta^2\|_D \leq \frac{C h^{r+1} t_{n-2}^{1-\alpha}}{\Gamma(2-\alpha)} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{C([0,T], H^{r+3})},$$

since  $b_j < (2-\alpha)j^{1-\alpha}$  from Lemma 3.1(ii). The estimate (5.28) also holds for  $n = 2$ , since  $b_0 = 1$ ,  $t_1 = \Delta t$ .

Since  $1 < \alpha < 2$ , we have  $1 + \alpha > 3 - \alpha$ . Therefore, substituting (5.24), (5.25), (5.27) and (5.28) in (5.23), we obtain

$$(5.29) \quad \left\| \delta_t \mathcal{F}_u^{n-\frac{1}{2}} \right\|_D \leq C (\Delta t^{3-\alpha} + h^{r+1}).$$

Also, it follows from (5.17), (5.18), and (5.29) that

$$(5.30) \quad \|\zeta^n\|_{H^2}^2 \leq C (\Delta t^{3-\alpha} + h^{r+1}),$$

since  $\zeta^0 = 0$ . Finally, applying the triangle inequality, (5.30) and Lemma 5.1 complete the proof.  $\square$

*Remark.* Note that results similar to those in [19, Remark 2.7] are also valid in this paper.

**6. Numerical experiments.** We present numerical results which support the analyses of preceding sections. In our implementations, we used the space of piecewise Hermite bicubics with the standard basis functions [8] on identical uniform partitions of  $\bar{T}$  in both the  $x$  and  $y$  directions with  $N_x = N_y = N$ . The initial conditions are approximated using the piecewise Hermite bicubic interpolant. We present  $L^\infty$ ,  $L^2$ ,  $H^1$  and  $H^2$  norms of the errors at  $T = 1$  and the corresponding rates of convergence determined by

$$(6.1) \quad \text{Rate} \approx \frac{\log(e_m/e_{m+1})}{\log(h_m/h_{m+1})},$$

where  $h = 1/N_m$  is the step size with  $N = N_m$ , and  $e_m$  is the norm of the corresponding error. The  $L^\infty$  norm of the error is estimated by calculating the maximum error at  $100 \times 100$  equally spaced points in each sub-rectangle  $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ ,  $1 \leq i, j \leq N$ . The  $H^\ell$  norm,  $\ell = 0, 1, 2$ , is computed using the ten-point composite Gauss quadrature rule so that the error due to quadrature does not affect the convergence rate.

**Example.** [28] We consider the problem (1.1)–(1.3) with  $T = 1$  and exact solution

$$u(x, y, t) = t^{2+\alpha} \sin(\pi x) \sin(\pi y), \quad (x, y) \in [0, 1] \times [0, 1], \quad t \in (0, T],$$

so that

$$\varphi(x, y) = 0, \quad \phi(x, y) = 0, \quad (x, y) \in [0, 1] \times [0, 1].$$

For the results in Tables 6.1–6.3, we selected the time step  $\Delta t = h^3$ , since, from our theoretical estimates, the error in the  $H^\ell$  norm,  $\ell = 0, 1, 2$ , is expected to be  $O(\Delta t^{3-\alpha} + h^{4-\ell})$  when  $r = 3$ . In Table 6.1, we present the  $L^2$  and  $L^\infty$  errors with their corresponding convergence rates which are seen to be approximately 4 as expected. Table 6.2 demonstrates the optimal convergence rates in the  $H^\ell$ ,  $\ell = 1, 2$ ,

TABLE 6.1  
 $L^2$  and  $L^\infty$  errors and convergence rates with  $\Delta t = h^3$ ,  $\alpha = 1.5$ .

| $N$ | $L^2$ error | Rate   | $L^\infty$ error | Rate   |
|-----|-------------|--------|------------------|--------|
| 4   | 3.1505e-004 |        | 1.1250e-3        |        |
| 6   | 5.3158e-5   | 4.3887 | 2.0968e-4        | 4.1433 |
| 9   | 1.0142e-5   | 4.0856 | 4.2644e-5        | 3.9281 |
| 12  | 3.1984e-5   | 4.0115 | 1.3239e-5        | 4.0660 |

TABLE 6.2  
 $H^1$  and  $H^2$  errors and convergence rates with  $\Delta t = h^3$ ,  $\alpha = 1.5$ .

| $N$ | $H^1$ error | Rate   | $H^2$ error | Rate   |
|-----|-------------|--------|-------------|--------|
| 4   | 6.2642e-3   |        | 1.6180e-1   |        |
| 6   | 1.8393e-3   | 3.0224 | 7.1581e-2   | 2.0113 |
| 9   | 5.4400e-4   | 3.0044 | 3.1744e-2   | 2.0054 |
| 12  | 2.2938e-4   | 3.0018 | 1.7844e-2   | 2.0023 |

norms, consistent with the theory. In Table 6.3, we present the maximum error in the approximations  $(U_h^M)_x, (U_h^M)_y$  to  $u_x, u_y$ , respectively, at the partition nodes, together with the corresponding convergence rate. From this table, we observe superconvergence, the rate of convergence being approximately 4 when  $r = 3$ . In Tables 6.4–6.6, we present the  $L^2$  and  $L^\infty$  errors and the temporal convergence rates, which is approximately  $(3 - \alpha)$  as expected.

**7. Concluding Remarks.** We have formulated and analyzed an ADI OSC Crank-Nicolson method for the two-dimensional fractional diffusion-wave equation. Under certain smoothness assumptions, we have proved that the method is of optimal global accuracy and exhibits superconvergence phenomena. The results of numerical experiments confirm the analysis.

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TABLE 6.3

Maximum nodal errors in  $(U_h^M)_x, (U_h^M)_y$  and convergence rates with  $\Delta t = h^3$ ,  $\alpha = 1.5$ .

| $N$ | Maximum nodal error in $(U_h^M)_x, (U_h^M)_y$ | Rate   |
|-----|---|--------|
| 4   | 1.1613e-3                                     |        |
| 6   | 3.2174e-4                                     | 3.1656 |
| 9   | 6.6230e-5                                     | 3.8983 |
| 12  | 2.1151e-5                                     | 3.9677 |

TABLE 6.4

 $L^2$  and  $L^\infty$  errors and convergence rates with  $\Delta t = h$ ,  $\alpha = 1.1$ .

| $N$ | $L^2$ error | Rate   | $L^\infty$ error | Rate   |
|-----|-------------|--------|------------------|--------|
| 40  | 1.3847e-6   |        | 2.7240e-6        |        |
| 80  | 4.4740e-7   | 1.6299 | 6.8463e-7        | 1.9923 |
| 160 | 1.3113e-7   | 1.7706 | 1.8874e-7        | 1.8589 |
| 320 | 3.7282e-8   | 1.8144 | 5.2931e-8        | 1.8342 |

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TABLE 6.5  
 $L^2$  and  $L^\infty$  errors and convergence rates with  $\Delta t = h$ ,  $\alpha = 1.45$ .

| $N$ | $L^2$ error | Rate   | $L^\infty$ error | Rate   |
|-----|-------------|--------|------------------|--------|
| 20  | 6.3816e-5   |        | 1.0336e-4        |        |
| 40  | 2.3412e-5   | 1.4467 | 3.3951e-5        | 1.6062 |
| 80  | 8.1811e-6   | 1.5169 | 1.1623e-5        | 1.5465 |
| 160 | 2.8253e-6   | 1.5339 | 3.9989e-6        | 1.5393 |

TABLE 6.6  
 $L^2$  and  $L^\infty$  errors and convergence rates with  $\Delta t = h$ ,  $\alpha = 1.8$ .

| $N$ | $L^2$ error | Rate   | $L^\infty$ error | Rate   |
|-----|-------------|--------|------------------|--------|
| 20  | 3.6625e-4   |        | 5.3143e-4        |        |
| 40  | 1.6144e-4   | 1.1818 | 2.2916e-4        | 1.2135 |
| 80  | 7.0458e-5   | 1.1962 | 9.9696e-5        | 1.2007 |
| 160 | 3.0695e-5   | 1.1988 | 4.3413e-5        | 1.1994 |

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## APPENDIX A: Proof of (4.28).

LEMMA 7.1. *There exist non-negative integers  $m, q$ ,  $1 \leq m \leq n$  and  $0 \leq q \leq n-1$ , such that*

$$(A.1) \quad \sum_{j=1}^n b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle \leq b_q \langle \phi, \Delta U_h^m - \Delta U_h^0 \rangle,$$

*Proof.* We prove the result by induction. When  $n = 1$ ,

$$(A.2) \quad b_0 \langle \phi, \Delta U_h^1 - \Delta U_h^0 \rangle \leq b_q \langle \phi, \Delta U_h^m - \Delta U_h^0 \rangle,$$

and  $q = 0, m = 1$ . Therefore, (A.1) holds for the case  $n = 1$ .

Now suppose that

$$(A.3) \quad \sum_{j=1}^n b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle \leq b_q \langle \phi, \Delta U_h^m - \Delta U_h^0 \rangle,$$

$$1 \leq m \leq n, \quad 0 \leq q \leq n-1,$$

Then

$$(A.4) \quad \begin{aligned} & \sum_{j=1}^{n+1} b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle \\ &= \sum_{j=1}^n b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle + b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \\ &\leq b_q \langle \phi, \Delta U_h^m - \Delta U_h^0 \rangle + b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle. \end{aligned}$$

on using the induction hypothesis (A.3).

1. First consider the case  $m = n$ .

(a) If  $\langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle \geq 0$  and  $\langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \geq 0$ , then

$$\begin{aligned} & b_q \langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle + b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \\ & \leq b_q \langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle + b_q \langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \\ & = b_q \langle \phi, \Delta U_h^{n+1} - \Delta U_h^0 \rangle, \quad 0 \leq q \leq n-1. \end{aligned}$$

(b) If  $\langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle \geq 0$  and  $\langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \leq 0$ , then

$$\begin{aligned} & b_q \langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle + b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \\ & \leq b_q \langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle, \quad 0 \leq q \leq n-1. \end{aligned}$$

(c) If  $\langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle \leq 0$  and  $\langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \geq 0$ , then

$$\begin{aligned} & b_q \langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle + b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \\ & \leq b_n \langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle + b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \\ & = b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^0 \rangle. \end{aligned}$$

(d) If  $\langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle \leq 0$  and  $\langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \leq 0$ , then

$$\begin{aligned} & b_q \langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle + b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \\ & \leq b_n \langle \phi, \Delta U_h^n - \Delta U_h^0 \rangle + b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \\ & = b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^0 \rangle. \end{aligned}$$

Therefore, in this case, (A.1) holds.

2. Now consider the case  $m < n$ . Recall from (A.4) that

$$\begin{aligned} \text{(A.5)} \quad & \sum_{j=1}^{n+1} b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle \\ & = \sum_{j=1}^n b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle + b_n \langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle. \end{aligned}$$

- (a) If  $\langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \leq 0$ , then, by using the induction hypothesis (A.3),

$$\sum_{j=1}^{n+1} b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle \leq b_q \langle \phi, \Delta U_h^m - \Delta U_h^0 \rangle, \\ 1 \leq m \leq n, \quad 0 \leq q \leq n-1.$$

- (b) If  $\langle \phi, \Delta U_h^{n+1} - \Delta U_h^n \rangle \geq 0$ , then, from (A.5),

$$\sum_{j=1}^{n+1} b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle \\ \leq \sum_{j=1}^{n-1} b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle + b_{n-1} \langle \phi, \Delta U_h^{n+1} - \Delta U_h^{n-1} \rangle,$$

since  $b_n < b_{n-1}$ .

- i. If  $\langle \phi, \Delta U_h^{n+1} - \Delta U_h^{n-1} \rangle \leq 0$ , then, by using the induction hypothesis (A.3),

$$\sum_{j=1}^{n+1} b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle \\ \leq b_q \langle \phi, \Delta U_h^m - \Delta U_h^0 \rangle, \quad 1 \leq m \leq n-1, \quad 0 \leq q \leq n-2.$$

- ii. If  $\langle \phi, \Delta U_h^{n+1} - \Delta U_h^{n-1} \rangle \geq 0$ , from (A.6), then

$$\sum_{j=1}^{n+1} b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle \\ \leq \sum_{j=1}^{n-2} b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle + b_{n-2} \langle \phi, \Delta U_h^{n+1} - \Delta U_h^{n-2} \rangle.$$

Repeating the above process, we obtain

$$\sum_{j=1}^{n+1} b_{j-1} \langle \phi, \Delta U_h^j - \Delta U_h^{j-1} \rangle \\ \leq b_q \langle \phi, \Delta U_h^m - \Delta U_h^0 \rangle, \quad 1 \leq m \leq n+1, \quad 0 \leq q \leq n,$$

which completes the proof of (A.1).  $\square$

**APPENDIX B: Proof of (5.24).** If the hypotheses of Theorem (5.3) are satisfied, we will prove that

$$\left\| \delta_t R_\alpha^{n-\frac{1}{2}} \right\|_D \leq C \Delta t^{3-\alpha}.$$

We set  $\tau = \Delta t$ ,  $v = u_t$ , and  $v^n$  for the approximation to  $v(x, y, t_n)$ . From (1.1), we have

$$(B.1) \quad \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left[ b_0 v^{n-\frac{1}{2}} - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) v^{j-\frac{1}{2}} - b_{n-1} v^0 \right] \\ = \Delta u^{n-\frac{1}{2}} + f^{n-\frac{1}{2}} - \left( \widetilde{R}_t^\alpha \right)^{n-\frac{1}{2}},$$

where

$$\begin{aligned} \left( \widetilde{R}_t^\alpha \right)^n &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\partial v(x, y, s)}{\partial s} \frac{ds}{(t_n - s)^{\alpha-1}} \\ &\quad - \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left[ b_0 v^n - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) v^j - b_{n-1} v^0 \right] \\ &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\partial v(x, y, s)}{\partial s} \frac{ds}{(t_n - s)^{\alpha-1}} \\ &\quad - \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{j=1}^n \frac{v^j - v^{j-1}}{\tau} \int_{t_{j-1}}^{t_j} (t_n - s)^{1-\alpha} ds \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left\{ \frac{\partial v(x, y, s)}{\partial s} - \frac{v^j - v^{j-1}}{\tau} \right\} \frac{ds}{(t_n - s)^{\alpha-1}}. \end{aligned}$$

Using Taylor's theorem with integral remainder, and interchanging the order of integration in the resulting equation, we obtain

$$(B.2) \quad \left( \widetilde{R}_t^\alpha \right)^n = \frac{1}{\Gamma(2-\alpha)\tau} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left[ \int_{t_{j-1}}^s \frac{\partial^2 v(x, y, t)}{\partial t^2} (t - t_{j-1}) dt \right. \\ \left. - \int_s^{t_j} \frac{\partial^2 v(x, y, t)}{\partial t^2} (t_j - t) dt \right] \frac{ds}{(t_n - s)^{\alpha-1}} \\ = \frac{1}{\Gamma(3-\alpha)} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left\{ (t_n - s)^{2-\alpha} - \left[ \frac{s - t_{j-1}}{\tau} (t_n - t_j)^{2-\alpha} \right. \right. \\ \left. \left. + \frac{t_j - s}{\tau} (t_n - t_{j-1})^{2-\alpha} \right] \right\} \frac{\partial^2 v}{\partial s^2}(x, y, s) ds \\ = \frac{1}{\Gamma(3-\alpha)} (R_t^\alpha)^n.$$

With  $g(s) = (t_n - s)^{2-\alpha}$ ,

$$(B.3) \quad (R_t^\alpha)^n = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left\{ g(s) - \left[ \frac{s - t_{k-1}}{\tau} g(t_k) + \frac{t_k - s}{\tau} g(t_{k-1}) \right] \right\} \frac{\partial^2 v}{\partial s^2}(x, y, s) ds \\ = \widehat{r}_q + r_q,$$



where

$$\begin{aligned}\widehat{r}_q &= \sum_{k=1}^{n-q} \int_{t_{k-1}}^{t_k} \left\{ g(s) - \left[ \frac{s-t_{k-1}}{\tau} g(t_k) + \frac{t_k-s}{\tau} g(t_{k-1}) \right] \right\} \frac{\partial^2 v}{\partial s^2}(x, y, s) ds \\ r_q &= \sum_{k=n-q+1}^n \int_{t_{k-1}}^{t_k} \left\{ g(s) - \left[ \frac{s-t_{k-1}}{\tau} g(t_k) + \frac{t_k-s}{\tau} g(t_{k-1}) \right] \right\} \frac{\partial^2 v}{\partial s^2}(x, y, s) ds.\end{aligned}$$

On applying Lemma 2.1 in [13], we have

$$\widehat{r}_q = \frac{(2-\alpha)(\alpha-1)}{2} \sum_{k=1}^{n-q} (t_n - \xi_k)^{-\alpha} \int_{t_{k-1}}^{t_k} (s - t_{k-1})(t_k - s) \frac{\partial^2 v}{\partial s^2}(x, y, s) ds,$$

where  $\xi_k \in (t_{k-1}, t_k)$ . Then

$$\begin{aligned}\delta_t \widehat{r}_q &= \frac{(2-\alpha)(\alpha-1)}{2} \sum_{k=1}^{n-q-1} \delta_t (t_n - \xi_k)^{-\alpha} \int_{t_{k-1}}^{t_k} (s - t_{k-1})(t_k - s) \frac{\partial^2 v}{\partial s^2}(x, y, s) ds \\ &+ \frac{(2-\alpha)(\alpha-1)}{2} \frac{(t_n - \xi_{n-q})^{-\alpha}}{\tau} \int_{t_{n-q-1}}^{t_{n-q}} (s - t_{n-q-1})(t_{n-q} - s) \frac{\partial^2 v}{\partial s^2}(x, y, s) ds,\end{aligned}$$

and since

$$\int_{t_{k-1}}^{t_k} (s - t_{k-1})(t_k - s) ds = \frac{\tau^3}{6},$$

we obtain

$$\begin{aligned}(\text{B.4}) \quad |\delta_t \widehat{r}_q| &\leq C\tau^3 \max_{t_0 \leq t \leq t_n} \left| \frac{\partial^2 v}{\partial t^2}(x, y, t) \right| \left| \sum_{k=1}^{n-q-1} (\widehat{\xi}_{n-1} - \xi_k)^{-\alpha-1} \right| \\ &+ C\tau^3 \max_{t_0 \leq t \leq t_n} \left| \frac{\partial^2 v}{\partial t^2}(x, y, t) \right| \frac{(t_n - t_{n-q})^{-\alpha}}{\tau} \\ &\leq C\tau^3 \max_{t_0 \leq t \leq t_n} \left| \frac{\partial^2 v}{\partial t^2}(x, y, t) \right| \left| \sum_{k=1}^{n-q-1} (t_{n-1} - t_k)^{-\alpha-1} \right| \\ &+ C\tau^2 t_q^{-\alpha} \max_{t_0 \leq t \leq t_n} \left| \frac{\partial^2 v}{\partial t^2}(x, y, t) \right| \\ &\leq C\tau^2 t_{q-1}^{-\alpha} \max_{t_0 \leq t \leq t_n} \left| \frac{\partial^2 v}{\partial t^2}(x, y, t) \right|,\end{aligned}$$

where  $\widehat{\xi}_{n-1} \in (t_{n-1}, t_n)$ ,  $\xi_k \in (t_{k-1}, t_k)$ .

To estimate  $r_q$ , we first note that, from Taylor's theorem, we have

$$\frac{\partial^2 v}{\partial s^2}(x, y, s) = \frac{\partial^2 v}{\partial s^2}(x, y, t_n) + (s - t_n) \frac{\partial^3 v}{\partial s^3}(x, y, \bar{\xi}_n), \quad \bar{\xi}_n \in (s, t_n),$$

so that

$$\begin{aligned} r_q &= \sum_{k=n-q+1}^n \int_{t_{k-1}}^{t_k} \left\{ g(s) - \left[ \frac{s-t_{k-1}}{\tau} g(t_k) + \frac{t_k-s}{\tau} g(t_{k-1}) \right] \right\} \frac{\partial^2 v}{\partial s^2}(x, y, t_n) ds \\ &+ \sum_{k=n-q+1}^n \int_{t_{k-1}}^{t_k} \left\{ g(s) - \left[ \frac{s-t_{k-1}}{\tau} g(t_k) + \frac{t_k-s}{\tau} g(t_{k-1}) \right] \right\} (s-t_n) \frac{\partial^3 v}{\partial s^3}(x, y, \bar{\xi}_n) ds. \end{aligned}$$

Since

$$\int_{t_{k-1}}^{t_k} \left( \frac{s-t_{k-1}}{\tau} g(t_k) + \frac{t_k-s}{\tau} g(t_{k-1}) \right) ds = \frac{\tau}{2} [g(t_k) + g(t_{k-1})],$$

we have

$$\begin{aligned} r_q &= \frac{\partial^2 v(x, y, t_n)}{\partial s^2} \left\{ \int_{t_{n-q}}^{t_n} g(s) ds - \sum_{k=n-q+1}^n \frac{\tau}{2} [g(t_k) + g(t_{k-1})] \right\} \\ &+ \sum_{k=n-q+1}^n \int_{t_{k-1}}^{t_k} \left\{ g(s) - \left[ \frac{s-t_{k-1}}{\tau} g(t_k) + \frac{t_k-s}{\tau} g(t_{k-1}) \right] \right\} (s-t_n) \frac{\partial^3 v}{\partial s^3}(x, y, \bar{\xi}_n) ds \\ &= \frac{\partial^2 v}{\partial s^2}(x, y, t_n) \left\{ \frac{q^{3-\alpha}}{3-\alpha} - \left[ \frac{q^{2-\alpha}}{2} + (q-1)^{2-\alpha} + (q-2)^{2-\alpha} + \dots + 1^{2-\alpha} \right] \right\} \tau^{3-\alpha} \\ &+ \sum_{k=n-q+1}^n \int_{t_{k-1}}^{t_k} \left\{ g(s) - \left[ \frac{s-t_{k-1}}{\tau} g(t_k) + \frac{t_k-s}{\tau} g(t_{k-1}) \right] \right\} (s-t_n) \frac{\partial^3 v}{\partial s^3}(x, y, \bar{\xi}_n) ds. \end{aligned}$$

Thus,

$$\begin{aligned} (B.5) \quad & |\delta_t r_q| \\ & \leq \left| \delta_t \frac{\partial^2 v}{\partial s^2}(x, y, t_n) \right| \left| \frac{q^{3-\alpha}}{3-\alpha} - \left[ \frac{q^{2-\alpha}}{2} + (q-1)^{2-\alpha} + (q-2)^{2-\alpha} + \dots + 1^{2-\alpha} \right] \right| \tau^{3-\alpha} \\ & + \frac{1}{\tau} \left| \sum_{k=n-q+1}^n \int_{t_{k-1}}^{t_k} \left\{ g(s) - \left[ \frac{s-t_{k-1}}{\tau} g(t_k) + \frac{t_k-s}{\tau} g(t_{k-1}) \right] \right\} (s-t_n) \frac{\partial^3 v}{\partial s^3}(\cdot, \bar{\xi}_n) ds \right. \\ & \quad \left. - \sum_{k=n-q}^{n-1} \int_{t_{k-1}}^{t_k} \left\{ g(s) - \left[ \frac{s-t_{k-1}}{\tau} g(t_k) + \frac{t_k-s}{\tau} g(t_{k-1}) \right] \right\} (s-t_{n-1}) \frac{\partial^3 v(\cdot, \bar{\xi}_{n-1})}{\partial s^3} ds \right| \\ & \leq C \max_{t_0 \leq t \leq t_n} \left| \frac{\partial^3 v}{\partial t^3}(x, y, t) \right| \tau^{3-\alpha}. \end{aligned}$$

By using the notation  $v^{n-\frac{1}{2}} = \frac{1}{2}(v^n + v^{n-1})$ ,  $\delta_t u^n = \frac{u^n - u^{n-1}}{\tau}$ , and the Lemma 2.2 in [28], we have

$$(B.6) \quad v^{n-\frac{1}{2}} = \delta_t u^n + (R_t)^{n-\frac{1}{2}}, \quad 1 \leq n \leq M,$$

where

(B.7)

$$(R_t)^{n-\frac{1}{2}} = \frac{\tau^2}{16} \int_0^1 \left[ \frac{\partial^3 u}{\partial t^3}(x, y, t_{n-\frac{1}{2}} + \frac{s\tau}{2}) + \frac{\partial^3 u}{\partial t^3}(x, y, t_{n-\frac{1}{2}} - \frac{s\tau}{2}) \right] (1-s^2) ds.$$

Substituting (B.6) into (B.1), then the truncation error in (3.1),  $R_\alpha^{n-\frac{1}{2}}$ , can be written as

(B.8)

$$\begin{aligned} R_\alpha^{n-\frac{1}{2}} &= (\widetilde{R}_t^\alpha)^{n-\frac{1}{2}} - \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left[ (R_t)^{n-\frac{1}{2}} - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j})(R_t)^{j-\frac{1}{2}} \right] \\ &= \frac{1}{\Gamma(3-\alpha)} \left\{ (R_t^\alpha)^{n-\frac{1}{2}} - \tau^{1-\alpha} \left[ (R_t)^{n-\frac{1}{2}} - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j})(R_t)^{j-\frac{1}{2}} \right] \right\}. \end{aligned}$$

Since  $v = u_t$ , then combining (B.3), (B.4) and (B.5), we have

$$\begin{aligned} (B.9) \quad |\delta_t(R_t^\alpha)^n| &\leq |\delta_t r_q| + |\delta_t \widehat{r}_q| \\ &\leq C\tau^2 t_{q-1}^{-\alpha} \max_{t_0 \leq t \leq t_n} \left| \frac{\partial^3 u}{\partial t^3}(x, y, t) \right| + C \max_{t_0 \leq t \leq t_n} \left| \frac{\partial^4 u}{\partial t^4}(x, y, t) \right| \tau^{3-\alpha}. \end{aligned}$$

For  $2 \leq n \leq M$ , we have

$$\begin{aligned} (B.10) \quad \delta_t R_\alpha^{n-\frac{1}{2}} &= \frac{1}{\Gamma(3-\alpha)} \left\{ \delta_t (R_t^\alpha)^{n-\frac{1}{2}} - \tau^{1-\alpha} \left[ \delta_t (R_t)^{n-\frac{1}{2}} \right. \right. \\ &\quad \left. \left. - \sum_{j=1}^{n-2} (b_{j-1} - b_j) \delta_t (R_t)^{n-j-\frac{1}{2}} - \frac{b_{n-2} - b_{n-1}}{\tau} (R_t)^{\frac{1}{2}} \right] \right\}, \end{aligned}$$

Thus, for  $n \geq 2$

$$\begin{aligned} (B.11) \quad \left\| \delta_t R_\alpha^{n-\frac{1}{2}} \right\|_D &\leq \frac{1}{\Gamma(3-\alpha)} \left\{ \left\| \delta_t (R_t^\alpha)^{n-\frac{1}{2}} \right\|_D + \tau^{1-\alpha} \left\| \delta_t (R_t)^{n-\frac{1}{2}} \right\|_D \right. \\ &\quad \left. + \tau^{1-\alpha} \left[ \sum_{j=1}^{n-2} (b_{j-1} - b_j) \left\| \delta_t (R_t)^{n-j-\frac{1}{2}} \right\|_D + \frac{b_{n-2} - b_{n-1}}{\tau} \left\| (R_t)^{\frac{1}{2}} \right\|_D \right] \right\}. \end{aligned}$$

From (B.7), we have

$$(B.12) \quad \left| (R_t)^{n-\frac{1}{2}} \right| \leq \frac{\tau^2}{12} \max_{0 \leq t \leq T} \left| \frac{\partial^3 u}{\partial t^3}(x, y, t) \right|.$$

Combining (B.7) and (B.12), we have

$$(B.13) \quad \left| \delta_t (R_t)^{n-\frac{1}{2}} \right| \leq \frac{\tau^2}{12} \max_{0 \leq t \leq T} \left| \frac{\partial^4 u}{\partial t^4}(x, y, t) \right|.$$

Since  $\sum_{j=1}^{n-2} (b_{j-1} - b_j) = 1 - b_{n-2} \leq 1$ , and  $(2 - \alpha)(j + 1)^{1-\alpha} < b_j < (2 - \alpha)j^{1-\alpha}$ , we have

$$\begin{aligned} b_{n-2} - b_{n-1} &< (2 - \alpha) [(n - 2)^{1-\alpha} - n^{1-\alpha}] \\ &= (2 - \alpha)(1 - \alpha) \int_n^{n-2} x^{-\alpha} dx \\ &\leq 2(2 - \alpha)(\alpha - 1)(n - 2)^{-\alpha}. \end{aligned}$$

Thus

$$(B.14) \quad \tau^{1-\alpha} \left[ \sum_{j=1}^{n-2} (b_{j-1} - b_j) \left| \delta_t(R_t)^{n-j-\frac{1}{2}} \right| \right] \leq \frac{\tau^{3-\alpha}}{12} \max_{0 \leq t \leq T} \left| \frac{\partial^4 u}{\partial t^4}(x, y, t) \right|,$$

and

$$(B.15) \quad \frac{b_{n-2} - b_{n-1}}{\tau^\alpha} \left| (R_t)^{\frac{1}{2}} \right| \leq \frac{\tau^2(2 - \alpha)(\alpha - 1)t_{n-2}^{-\alpha}}{6} \max_{0 \leq t \leq T} \left| \frac{\partial^4 u}{\partial t^4}(x, y, t) \right|.$$

Combining (B.9), (B.10), (B.13), (B.14) and (B.15), we obtain

$$(B.16) \quad \left| \delta_t R_\alpha^{n-\frac{1}{2}} \right| \leq C \left\{ \tau^2 t_{q-1}^{-\alpha} \max_{t_0 \leq t \leq t_n} \left| \frac{\partial^3 u}{\partial t^3}(x, y, t) \right| + \tau^{3-\alpha} \max_{0 \leq t \leq T} \left| \frac{\partial^4 u}{\partial t^4}(x, y, t) \right| \right. \\ \left. + \frac{\tau^2(2 - \alpha)(\alpha - 1)t_{n-2}^{-\alpha}}{6} \max_{0 \leq t \leq T} \left| \frac{\partial^4 u}{\partial t^4}(x, y, t) \right| \right\}.$$

Since  $1 < \alpha < 2$ , we have  $2 > 3 - \alpha$ . Therefore, using the (B.16), we obtain

$$\left\| \delta_t R_\alpha^{n-\frac{1}{2}} \right\|_D \leq C \tau^{3-\alpha},$$

as desired.